The A-integral and Ahlfors-Beurling transform

Let Ω be a bounded domain in the complex plane. The Ahlfors-Beurling transform of a function $f \in L_p(\Omega)$, $1 \le p < \infty$ is defined as the following singular integral:

$$(B_\Omega f)(z) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\{w \in \Omega: |z-w| > \varepsilon\}} \frac{f(w)}{(z-w)^2} dm(w), \quad z \in \Omega.$$

The Ahlfors–Beurling transform is one of the important operators in complex analysis. It is the "Hilbert transform" on complex plane. This transform plays an essential role in applications to the theory of quasiconformal mappings and to the Beltrami equation with discontinuous coefficients. From the theory of singular integrals it is known that the Ahlfors–Beurling transform is a bounded operator in the space $L_p(\Omega)$, $1 < p < \infty$, that is, if $f \in L_p(\Omega)$, then $B_\Omega(f) \in L_p(\Omega)$ and the inequality

$$\|B_\Omega f\|_{L_p} \le C_p \|f\|_{L_p} \tag{1}$$

holds, where C_p is a constant independent of f. Besides if $f \in L_p(\Omega)$, $p > 1$ and $g \in L_q(\Omega)$, $q > 1$, $1/p + 1/q = 1$, then

$$\int_\Omega g(z)(B_\Omega f)(z) dm(z) = \int_\Omega f(z)(B_\Omega g)(z) dm(z). \tag{2}$$

In the case $f \in L(\Omega)$ only the weak inequality holds:

$$m\{z \in \Omega : |(B_\Omega f)(z)| > \lambda\} \le \frac{C_1}{\lambda} \|f\|_{L_1}, \quad \lambda > 0, \tag{3}$$

where m stands for the Lebesgue measure, C_1 is a constant independent of f. From the inequalities (1) and (3) it follows that the Ahlfors–Beurling transform of the function $f \in L(\Omega)$ satisfies the condition

$$m\{z \in \Omega : |(B_\Omega f)(z)| > \lambda\} = o\left(\frac{1}{\lambda}\right), \quad \lambda \to +\infty.$$

Note that the Ahlfors–Beurling transform of a function $f \in L(\Omega)$ is not Lebesgue integrable. In this paper we prove that the Ahlfors–Beurling transform of a function $f \in L(\Omega)$ is A-integrable on Ω and the analogue of equality (2) holds.

Theorem 1. Let $f \in L(\Omega)$ and $g(z)$ is a bounded function on Ω such that the $(B_\Omega g)(z)$ is also bounded on Ω. Then the function $g(z) \cdot (B_\Omega f)(z)$ is A-integrable on Ω and the equation

$$(A)\int_\Omega g(z)(B_\Omega f)(z) dm(z) = \int_\Omega f(z)(B_\Omega g)(z) dm(z)$$

holds.

Corollary 1. If $f \in L(\Omega)$ and the boundary of the domain Ω is a Lyapunov curve, then the function $(B_\Omega f)(z)$ is A-integrable on Ω.